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Collective Modes in Ising Lattices

J. K. Percus[1]

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The effect of collective modes on the otherwise local structure of Ising lattices is investigated by studying a number of exactly solvable models. First, the open one-dimensional Ising model serves to define sharp locality. This feature then remains upon extension to a Bethe lattice, despite the existence of a phase transition. But insertion of periodic boundary conditions creates a collective mode which breaks locality in a very specific fashion. A model interface is analyzed to show that even when locality is not broken, local uniformity can become untenable.

KEY WORDS: Ising lattice; Bethe lattice; density functional; collective mode; density profile; local correlations.

1. INTRODUCTION

There are two ancient diametrically opposed descriptions of the dynamics of a crystal lattice. In the Einstein model, the atoms—or molecules—at the various sites oscillate independently, a highly localized dynamical description. In the Debye model, each site is harmonically bound to its neighbors, the independent excitations now consisting of system-wide waves, collective oscillations of the whole system. Nonidealized systems will of course have both behavioral aspects. Since the local picture gives rise to simple and widely used approximate treatments of equilibrium properties, one would like to investigate under what circumstances the explicit insertion of collective modes is mandatory for accurate thermodynamics, and, more importantly, what form this insertion must take.

One way of distinguishing the degree of nonlocality is by considering nonuniform versions of the system being studied—nonuniform density for

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[1] Courant Institute of Mathematical Sciences and Physics Department, New York University, New York, New York 10012.

a particle system, or nonuniform magnetization for phenomena describable by spin. For simplicity, we will choose the latter, and so our domain is that of Ising models specified by the spins $\{\sigma_x\}$ at the lattice sites $\{x\}$, and the associated magnetization profile $m_x = \langle\sigma_x\rangle$. The nonuniformity being studied can be externally incited or self-maintained as in a phase separation, but in either case, an effective vehicle for the study is the Helmholtz free energy as a function of the magnetization profile. This implies that we are working with the *inverse* formulation (see, e.g., ref. 1) of the profile problem: for an external field h_x, with intersite interaction energy $\phi(\cdots\sigma_x\cdots)$, the grand potential at reciprocal temperature β and chemical potential μ is given by

$$\Omega = -\frac{1}{\beta}\ln \sum_{\{\sigma_x = \pm 1\}} \exp[-\beta\phi(\cdots\sigma_x\cdots) + \beta \sum_x (\mu - h_x)\,\sigma_x\} \tag{1.1}$$

so that

$$m_x = \partial\Omega/\partial h_x \tag{1.2}$$

In the inverse formulation, the m_x are chosen as the independent controlling variables, via the Legendre transform to the "internal" free energy

$$\begin{aligned}\bar{F}\{\cdots m_x\cdots\} &= \Omega + \sum_x (\mu - h_x)\, m_x \\ &= F - \sum m_x h_x\end{aligned} \tag{1.3}$$

and the profile equation takes the form

$$\mu - h_x = \partial\bar{F}/\partial m_x \tag{1.4}$$

If our system behaved *strictly locally*, this would manifest itself as

$$\bar{F} = \sum_x f(m_x) \tag{1.5}$$

where $f(\sigma)$ is the local free energy density, i.e., corresponding to the uniform bulk system, and consequently

$$\mu - h_x = f'(m_x) \tag{1.6}$$

would be strictly local as well. More generally, one could have *sharply local* entities extending over a few sites, e.g.,

$$\bar{F} = \tfrac{1}{2} \sum_{\langle x, y\rangle} f_2(m_x, m_y) + \sum_x f_1(m_x) \tag{1.7}$$

where $\langle x, y \rangle$ denotes nearest neighbor relation of x and y, and then

$$\mu - h_x = \sum_{\langle y, x \rangle}^{y} \frac{\partial}{\partial m_x} f_2(m_x, m_y) + f_1'(m_x) \tag{1.8}$$

is influenced only by nearby m_y. Alternatively, one can say that the generalized direct correlation function (for a survey see ref. 2)

$$\begin{aligned} C_2(x, y) &\equiv \partial \beta(\mu - h_x)/\partial m_y \\ &= \beta\, \partial^2 f_2(m_x, m_y)/\partial m_x\, \partial m_y\, \delta_{\langle y, x \rangle} \\ &\quad + \beta \left[f_1''(m_x) + \sum_{\langle z, x \rangle}^{z} \partial^2 f_2(m_x, m_z)/\partial m_x^2 \right] \delta_{y,x} \end{aligned} \tag{1.9}$$

has finite range. Finally, one may have mere *locality*, in which contributions to $\mu - h_x$ decrease sufficiently rapidly as one moves away from x, mirrored in obvious fashion by $\bar{F}$ and C_2.

In the presence of collective modes, e.g., ones originating dynamically as system-wide waves, vortices, surface waves on an interface,..., a local description may not make sense. One might imagine fixing the collective amplitudes, with the constrained ensemble being a superposition of locally uniform regions, but relaxation of the constraints would then introduce a superensemble of structures, killing locality, uniformity, or both, in each region. My intention in this paper is to use a sequence of solvable models to probe, in a preliminary fashion, when such nonlocal or *global* modes are to be anticipated, and the modifications in such as the free energy that they entail. This is of course all in the context of thermal equilibrium, but the dynamical consequences hovering in the background will be evident.

2. MEAN FIELD FORMAT

One approach to seeing why a sharply local free energy is a good first guess, and how to correct it, is by a neat reformulation due to Kac and Siegert,$^{(3)}$ among others. Consider the basic ferromagnetic nearest-neighbor coupling Ising model in an external field

$$\phi - M\mu = \sum_x (h_x - \mu)\, \sigma_x - \tfrac{1}{2} \left(\sum_{\langle x, y \rangle}^{xy} \sigma_x \sigma_y + J' \sum_x \sigma_x^2 \right) \tag{2.1}$$

The term $J' \sum \sigma_x^2 = NJ'$ for N sites is an irrelevant but very useful constant, being chosen so that the matrix

$$J_{xy} = J\delta_{\langle x, y \rangle} + J'\delta_{x,y} \tag{2.2}$$

is positive definite. The Kac–Siegert trick is then to observe that the energy exponent of the partition function

$$Z=\sum_{\{\sigma_x\}} \exp\left[\beta \sum_x (\mu-h_x)\,\sigma_x\right] \exp\left[\tfrac{1}{2}\beta \sum_{x,y} J_{xy}\sigma_x\sigma_y\right] \tag{2.3}$$

can be made fully linear in the $\{\sigma_x\}$ via the identity

$$\begin{aligned}&\left(\exp \tfrac{1}{2}\beta \sum J_{xy}\sigma_x\sigma_y\right)\\ &\quad=(\operatorname{Det} 2\pi J/\beta)^{-1/2}\\ &\qquad\times\int\cdots\int \exp\left(\beta \sum v_x\sigma_x\right)\exp\left(-\tfrac{1}{2}\beta \sum J_{xy}^{-1} v_x v_y\right)\prod dv_x\end{aligned} \tag{2.4}$$

Then the σ_x summations can be done explicitly,

$$\sum_{\sigma_x=\pm 1} \exp[\beta(v_x+\mu-h_x)\,\sigma_x]=2\cosh \beta(v_x+\mu-h_x) \tag{2.5}$$

resulting in the "continuous spin" form

$$\begin{aligned}Z&=(\operatorname{Det} 2\pi J/\beta)^{-1/2}\int\cdots\int \exp\left(-\beta/2 \sum J_{xy}^{-1} v_x v_y\right)\\ &\quad\times\prod_x 2\cosh \beta(v_x+\mu-h_x)\prod dv_x\end{aligned} \tag{2.6}$$

What is so special and convenient about (2.6)? If J^{-1} is "large" (e.g., J long-range and weak), a standard steepest descent expansion should work, with the integral being dominated by the vicinity of one point in v space; let us take this as an approximation. At the point in question, $\bar{v}$, the gradient of the integrand of (2.6) vanishes:

$$\sum_y J_{xy}^{-1}\bar{v}_y=\tanh \beta(\bar{v}_x+\mu-h_x) \tag{2.7}$$

[the maximizing point $\bar{v}$ being chosen when (2.7) has more than one solution]. But also, if W denotes the integrand—the unnormalized probability in v space—then if $\bar{v}$ is dominant,

$$\begin{aligned}m_x=-\frac{1}{\beta}\frac{\partial Z/\partial h_x}{Z}&=\frac{\int W \tanh \beta(v_x+\mu-h_x)\prod dv_x}{\int W\prod dv_x}\\ &=\tanh \beta(\bar{v}_x+\mu-h_x)\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have

$$\bar{v}_x = \sum_y J_{xy} m_y \tag{2.9}$$

so that $\bar{v}_x$ is recognized as the "mean field" due to internal forces (see, e.g., ref. 4).

The thermodynamic analysis of (2.1) is now immediate. We have from (2.7) and (2.8) $\mu - h_x = (1/\beta)(\tanh^{-1} \sum J_{xy}^{-1} \bar{v}_y - \bar{v}_x)$, and hence from (2.9)

$$\mu - h_x = \frac{1}{\beta} \tanh^{-1} m_x - \sum J_{xy} m_y \tag{2.10}$$

which is indeed a sharply local relationship. Furthermore, we can integrate back, via (1.4), to obtain the corresponding local free energy

$$\bar{F} = \sum_x f(m_x) - \tfrac{1}{2} \sum J_{xy} m_x m_y \tag{2.11}$$

where

$$f(z) = \frac{1}{\beta} \int \tanh^{-1} z \, dz$$

The locality of (2.10) and (2.11) does not preclude the expected phase transition at some β_c, signalled by the appearance of two solutions bifurcating from the uniform $\mu - h_x = 0$, $m_x = 0$ solution at low β, and occurring at the β_c at which

$$\frac{\partial \mu - h_x}{\partial m_y} = \frac{1}{\beta_c} \frac{1}{1 - m_x^2} \delta_{x,y} - J_{xy} \tag{2.12}$$

first becomes singular for $m_x = 0$. At β_c, there is also a nonuniform two-phase solution to (2.10) with $\mu - h_x = 0$ and long-range correlations along the two-phase interface. But $\bar{F}$ of (2.11) is a strange free energy—it does not arise exactly from any interaction energy, and despite the long-range correlations in (2.10) under interface conditions, there is no softening of the profile with increasing volume of the lattice.

There is little difficulty in carrying out corrections to steepest descent, picking up the approximate interfacial surface modes, and correcting the evaluation of (2.6) (see, e.g., ref. 5). But (2.6) is not strictly or even sharply local at zeroth order; extending the corrections to $\bar{F}$, in which nonlocality would be meaningful, is a difficult job. We must deal with something simpler than the general expression (2.1).

3. ONE-DIMENSIONAL LATTICE GAS

To get some feeling as to when and how collective modes play a role, it would be nice to start with a system in which they are explicitly—manifestly—absent, a *rigorously* local nontrivial model. Indeed, such a model exists, the one-dimensional nearest neighbor Ising model,(6) whose extensions, trivial and nontrivial, will serve as our major tools. Technically, it will be a bit more convenient to regard this as a lattice gas, and with no increase of difficulty to imagine the occupation number v_x to take the values 0, 1,..., D at each site. The site, and adjacent site, Boltzmann factors will now be denoted by

$$w_x(v) = e^{\beta(\mu - u_x)v}, \qquad e(v, v') = e^{-\beta\phi(v,v')} \tag{3.1}$$

where u_x is the external potential and the site–site potential $\phi(v, v')$ need not be specified yet [nor is the specific form of $w_x(v)$ in (3.1) required]. The profile equation we seek is the relation between $\mu - u_x$ and the "density"

$$n_y(v) = \langle \delta_{v_y, v} \rangle = \mathrm{Prob}(v_y = v) \tag{3.2}$$

Now, on the infinite line, we have

$$Z = \sum_{\{v_x\}} \prod_x w_x(v_x) \prod_x e(v_x, v_{x+1}) \tag{3.3}$$

Furthermore, $n_y(v)$, obtained by fixing v_y, decomposes (3.3) into three contributions, namely

$$n_y(v) = w_y(v)\, Z_y(v)\, \hat{Z}_y(v)/Z \tag{3.4}$$

in terms of the truncated partition functions

$$Z_y(v) = \sum_{\{v_x\}} \left(\prod_{x=-\infty}^{y-1} w_x(v_x) \left[\prod_{-\infty}^{y-1} e(v_x, v_{x-1}) \right] e(v, v_{y-1}) \right) \tag{3.5}$$

$$\hat{Z}_y(v) = \sum_{\{v_x\}} \left[\prod_{x=y+1}^{\infty} w_x(v_x) \right] \left[\prod_{y+2}^{\infty} e(v_x, v_{x-1}) \right] e(v_{y+1}, v) \tag{3.6}$$

summation being only over the arguments present. Equations (3.5) and (3.6) will be replaced by the simple recursion relations

$$Z_y(v) = \sum_{v'} e(v, v')\, w_{y-1}(v')\, Z_{y-1}(v') \tag{3.7}$$

$$\hat{Z}_y(v) = \sum_{v'} \hat{Z}_{y+1}(v')\, w_{y+1}(v')\, e(v', v) \tag{3.8}$$

By eliminating $w_y(v)$ from (3.7) and (3.8) via (3.4) and combining the results, we readily arrive at the pair

$$\sum_{v', v''} e^{-1}(v, v')\, e(v, v'')\, n_y(v'')\, Z_y(v')/Z_y(v'') = n_{y-1}(v) \tag{3.9}$$

$$\sum_{v', v''} e^{-1}(v, v')\, e(v, v'')\, n_y(v'')\, \hat{Z}_y(v')/\hat{Z}_y(v'') = n_{y+1}(v) \tag{3.10}$$

both of which take the form

$$\sum_{v', v''} e^{-1}(v, v')\, e(v, v'')\, n(v'')\, R(v', v'', n, n') = n'(v) \tag{3.11}$$

where

$$R(v', v'', n, n') = G(v', n, n')/G(v'', n, n')$$

subsequent to which the profile equation is given by (3.4) as

$$\frac{w_y(v'')}{w_y(v')} = R(v', v'', n_y, n_{y-1})\, R(v, v'', n_y, n_{y+1})\, \frac{n_y(v'')}{n_y(v')} \tag{3.12}$$

Specializing now to the ferromagnetic Ising case, in which $v = 0, 1$, the gas density is $n_x = n_x(1)$, while $n_x(0) = 1 - n_x$; we have $w_x(0) = 1$, $w_x(1) = e^{\beta(\mu - u_x)}$, $\phi(v, v') = -Jvv'$. Then Eq. (3.11) reduces to a single equation for $R(0, 1, n, n') \equiv R(n, n')$ (by choosing $v = 0$)

$$en(1)\, R - n(0)\, \frac{1}{R} + en(0) - n(1) + n'(0)\, f = 0 \tag{3.13}$$

where $e = e^{J\beta}$, $f = e - 1$, so that

$$\begin{aligned} 2R(n, n') = &-\left[\frac{1 + f'n}{en} - \left(1 + \frac{1}{e}\right)\right] \\ &+ 1 \left\{\left[\frac{1 + f'n}{en} - \left(1 + \frac{1}{e}\right)\right]^2 + \frac{4}{e}\, \frac{1 - n}{n}\right\}^{1/2} \end{aligned} \tag{3.14}$$

Since (3.12) now reads

$$\beta(\mu - u_x) = \ln \frac{n_x}{1 - n_x} + \ln R(n_x, n_{x-1}) + \ln R(n_x, n_{x+1}) \tag{3.15}$$

we conclude that the profile equation is indeed sharply local, as is the free energy $\bar{F}$, although this is a somewhat more complicated object.

4. THE BETHE LATTICE

Now that we have a model with a bona fide local structure, we can try to bring in collective modes. Enforcing a phase separation, a weakly maintained spatial boundary, is an obvious possibility, most easily invoked by locating a bifurcation of the uniform density. This suggests working on a Bethe lattice of coordination number c (equivalent to c rooted $(c-1)$-branch Cayley trees from a common origin), the context in which—subject to a number of caveats$^{(7)}$ relating to the domination of such a lattice by its surface—the Bethe–Peierls phase transition$^{(8)}$ becomes exact. We can proceed very much as in the one-dimensional, or $c=2$, case (see also ref. 9). Here, (3.3) must be replaced by

$$Z=\sum_{\{v_x\}} \prod_x w_x(v_x) \prod_{\langle x,\, y\rangle} e(v_x, v_y) \tag{4.1}$$

where the product is over unordered nearest neighbor pairs $\langle x, y\rangle$, while (3.4) decomposes instead into

$$n_y(v)=w_y(v) \prod_{i=1}^{c} Z_y^{(i)}(v)/Z \tag{4.2}$$

$Z_y^{(i)}(v)$ is the contribution of the direction-i Cayley tree rooted at y, given in detail as

$$Z_y^{(i)}(v)=\sum_{\{v_x \mid y+\hat{\imath} \in [y,x]\}} e(v, v_{y+\hat{\imath}}) \left[\prod_x w_x(v_x)\right] \prod_{\langle x,\, y\rangle} e(v_x, v_y) \tag{4.3}$$

where $\hat{\imath}$ signifies a vectorial bond in direction i.

Again, we need the recursion relation satisfied by $Z_y^{(i)}(v)$. Assuming for convenience that c is even, this is readily verified to be

$$Z_y^{(i)}(v)=\sum_{v'} e(v, v') \left[\prod_{j \neq -i} Z_{y+\hat{\imath}}^{(j)}(v')\right] w_{y+\hat{\imath}}(v') \tag{4.4}$$

where $-i$ is the direction opposite i. Equation (4.4) leads via (4.2) to

$$Z_y^{(i)}(v)=\sum_{v'} e(v, v') \frac{n_{y+\hat{\imath}}(v)}{Z_{y+\hat{\imath}}^{(-i)}(v')} Z \tag{4.5}$$

and hence to

$$n_y(v)/Z_y^{(i)}(v)=\sum_{v'} e^{-1}(v, v')\, Z_{y+\hat{\imath}}^{(-i)}(v')/Z \tag{4.6}$$

Combining (4.5) and (4.6), we find the desired

$$\sum_{v,v'} e^{-1}(v, v')\, e(v, v'')\, \frac{Z_y^{(i)}(v')}{Z_y^{(i)}(v'')}\, n_y(v'') = n_{y+i}(v) \tag{4.7}$$

Since (4.7) is again of the form (3.11), the computation, e.g., (3.13) of the one-dimensional lattice can be taken over intact. The only difference is that (4.2), in the form

$$\frac{w_y(v'')}{w_y(v')} = \frac{n_y(v'')}{n_y(v')} \prod_i \frac{Z_y^{(i)}(v')}{Z_y^{(i)}(v'')} \tag{4.8}$$

now determines the resulting profile. In particular, in the Ising case,

$$\beta(\mu - u_x) = \ln \frac{n_x}{1 - n_x} + \sum_{i=1}^{c} \ln R(0, 1, n_x, n_{x+i}) \tag{4.9}$$

Equation (4.9) of course is sharply local. But the uniform $m = 0$ or $n = 1/2$ case does bifurcate at the Bethe–Peierls transition temperature: Setting $u_x = 0$, $n_x = n$, (4.9) becomes

$$\beta\mu = \ln \frac{n}{1 - n} - c \ln 2ne + c \ln\{2n - 1 + [(2n - 1)^2 + 4en(1 - n)]^{1/2}\} \tag{4.10}$$

so that

$$\left.\frac{\partial \beta\mu}{\partial n}\right|_{n = 1/2} = 4 - 2c + 2ce^{-1/2} \tag{4.11}$$

which vanishes at the classical Bethe–Peierls

$$J\beta_c = 2 \ln \frac{c}{2 - c} \tag{4.12}$$

Clearly the phase transition and implied phase separation do not produce the global modes needed to destroy locality.

5. GLOBAL MODES

In retrospect, the failure of the Bethe lattice to invoke global nonlocal behavior, even in the presence of some sort of phase transition, is not surprising. One is seeking cooperative effects, in which a site x which influences y is in turn influenced by y *through another path.* In other words, simple connectivity will not do. If this is really the case, we must allow for large-scale interaction circuits, and this can be done even in the one-dimensional Ising model by imposing periodic boundary conditions.[10] Let us do so, returning now to the spin representation $\{\sigma_x\}$, $x = 1,..., N$, with σ_{N+1} identified with σ_1. We thus have

$$Z = \sum_{\{\sigma_x\}} \prod_{x=1}^{N} w_x(\sigma_x)\, e(\sigma_x, \sigma_{x+1}) \tag{5.1}$$

where

$$w_x(\sigma) = e^{\beta(\mu - h_x)\sigma}, \qquad e(v, v') = e^{\beta J v v'}, \qquad \sigma_{N+1} = \sigma_1$$

or in obvious matrix notation

$$Z = \mathrm{Tr}(w_1 e w_2 e \cdots w_N e) \tag{5.2}$$

Instead of two truncated partition functions, unavailable because of the closure of the ring, we introduce first a normalized bond-excised partition function, the matrix

$$\zeta_{x,x+1} = e w_{x+2} e w_{x+3} \cdots w_{x-2} e w_{x-1} e/Z \tag{5.3}$$

and then the corresponding site-excised

$$\zeta_x = e w_{x+1} e w_{x+2} \cdots w_{x-2} e w_{x-1} e/Z \tag{5.4}$$

Clearly

$$n_x = \mathrm{Tr}\, \sigma \zeta_x w_x, \qquad 1 = \mathrm{Tr}\, \zeta_x w_x \tag{5.5}$$

where σ is the diagonal matrix $\sigma\delta_{\sigma,\sigma'}$. Since $\zeta_x = e w_{x+1} \zeta_{x,x+1}$, while $\zeta_{x+1} = \zeta_{x,x+1} w_x e$, we have the basic recursive relation

$$e w_{x+1} \zeta_{x+1} = \zeta_x w_x e \tag{5.6}$$

An immediate consequence (note that Det $w_x = 1$) is that

$$\mathrm{Det}\, \zeta_{x+1} = \mathrm{Det}\, \zeta_x = K \tag{5.7}$$

where K is a site-independent function of the σ_x.

We now want to eliminate the ζ_x from (5.5) and (5.6). To start with, Eqs. (5.5) imply that

$$\begin{aligned} \zeta_x(-1, -1)\, w_x(-1) &= \tfrac{1}{2}(1 - m_x) \\ \zeta_x(1, 1)\, w_x(1) &= \tfrac{1}{2}(1 + m_x) \end{aligned} \tag{5.8}$$

In order to find the $(-1, 1)$ and $(1, -1)$ matrix elements, we use (5.6) to write

$$\sigma w_{x+1}\zeta_{x+1} = \sigma e^{-1}\zeta_x w_x e \tag{5.9}$$

and take the trace to obtain

$$-\zeta_x(-1, 1)\, w_x(1) + \zeta_x(1, -1)\, w_x(-1) = cm_x - sm_{x+1} \equiv a_x \tag{5.10}$$

where $c = \cosh 2\beta J$, $s = \sinh 2\beta J$. Similarly, writing (5.6) as

$$\sigma\zeta_{x-1} w_{x-1} = \sigma e w_x J_x e^{-1} \tag{5.11}$$

and taking the trace, one gets

$$\zeta_x(-1, 1)\, w_x(-1) - \zeta_x(1, -1)\, w_x(1) = cm_x - sm_{x-1} \equiv b_x \tag{5.12}$$

Solving (5.11) and (5.12), and setting $w_x(1) = w_x$, $w_x(-1) = 1/w_x$, one has

$$\begin{aligned} \zeta_x(-1, 1) &= (b_x/w_x + a_x w_x)/(w_x^2 - w_x^{-2}) \\ \zeta_x(1, -1) &= (b_x w_x + a_x/w_x)/(w_x^2 - w_x^{-2}) \end{aligned} \tag{5.13}$$

Hence, condition (5.7) becomes

$$\tfrac{1}{4}(1 - m_x^2) - [a_x^2 + b_x^2 + a_x b_x(w_x^2 + w_x^{-2})]/(w_x^2 - w_x^{-2}) = K \tag{5.14}$$

which we solve as

$$\beta(\mu - h_x) = \tfrac{1}{2} \ln[-b_x + (b_x^2 - m_x^2 + 1 - 4K)^{1/2}]/[a_x + (a_x^2 - m_x^2 + 1 - 4K)^{1/2}] \tag{5.15}$$

The reduction of our system to (5.6) has made the functional form of K unavailable. Before showing how to find it, the full set of consequences of its existence can be ascertained. The important observation is that, by direct computation,

$$\left.\frac{\partial}{\partial m_y}(\mu - h_x)\right|_K = \left.\frac{\partial}{\partial m_x}(\mu - h_y)\right|_K \tag{5.16}$$

It follows that there is a free energy on the expanded space of $\mathbf{m} = \{m_x\}$ and K such that

$$\mu - h_x = \partial F_K(\mathbf{m}, K)/\partial m_x|_K \tag{5.17}$$

But we also know that there is an $\bar{F}(\mathbf{m})$ for which

$$\mu - h_x = \partial \bar{F}(\mathbf{m})/\partial m_x \tag{5.18}$$

Hence

$$\partial \bar{F}(\mathbf{m})/\partial m_x = \partial F_K(\mathbf{m}, K(\mathbf{m}))/\partial m_x - \partial K/\partial m_x \, \partial F_K(\mathbf{m}, K)/\partial K|_{K=K(\mathbf{m})}$$

or if

$$\Delta(\mathbf{m}) = \bar{F}(\mathbf{m}) - F_K(\mathbf{m}, K(\mathbf{m})) \tag{5.19}$$

then

$$\partial F_K/\partial K \, \partial K/\partial m_x = -\partial \Delta/\partial m_x \tag{5.20}$$

(5.20) implies that

$$\Delta(\mathbf{m}) = W(K(\mathbf{m})) \tag{5.21}$$

for some function W, and that

$$\bar{F}(\mathbf{m}) = F_K(\mathbf{m}, K(\mathbf{m})) + W(K(\mathbf{m})) \tag{5.22}$$

From (5.20) and (5.21), we see that

$$\partial F_K(\mathbf{m}, K)/\partial K|_{K=K(\mathbf{m})} = -\partial W(K)/\partial K|_{K=K(\mathbf{m})} \tag{5.23}$$

We conclude therefore, on defining

$$\bar{F}(\mathbf{m}, K) = F_K(\mathbf{m}, K) + W(K) \tag{5.24}$$

that at $K = K(\mathbf{m})$,

$$\begin{aligned} \bar{F}(\mathbf{m}, K) &= \bar{F}(\mathbf{m}) \\ \mu - h_x &= \partial \bar{F}(\mathbf{m}, K)/\partial m_x \\ 0 &= \partial \bar{F}(\mathbf{m}, K)/\partial K \end{aligned} \tag{5.25}$$

That is, the expanded space free energy $\bar{F}(\mathbf{m}, K)$ reduces to $\bar{F}(\mathbf{m})$, and $\mu - h_x$ is conjugate to m_x, 0 to K on the expanded space, where K is given its explicit form $K(\mathbf{m})$.

To actually find $K(\mathbf{m})$, we need $W(K)$, and for this it is sufficient to use (5.24) at uniformity, $m_x = m$, an easy computation. Interestingly, as well, K is related to the grand partition function

$$K = 2^N s^N / Z^2 \tag{5.26}$$

showing both its global character and the fact that $K \to 0$ as $N \to \infty$.

6. BREAKING OF LOCAL UNIFORMITY

One aspect of global modes is clear. The system can remain formally local, but parametrized by global amplitudes. This, for example, can also be the case when a locally describable system is augmented by an interaction potential of low rank as a matrix. However, the idealization of strict locality implies that of local uniformity, which may make no sense if each site requires an ensemble of locally uniform states. Another extension of our basic one-dimensional Ising model(11) illustrates this point. It is that of a square lattice of integer coordinates (x, t), with a nominally vertical interface—between "vacuum" on the left and "liquid" of density 1 on the right—composed of upwardly directed bonds at ±45° from the vertical. A configuration can be described either by spins $\{\sigma_t = \pm 1\}$ for a ±45° direction leaving a site of altitude t, or by $\{x_t\}$, the horizontal locations of the ends of the bonds, or for that matter by the pairs (x_t, σ_t). In any event, this is effectively a one-dimensional lattice, with a "spin" degree of freedom at each site. The "interfacial" energy can be taken as

$$\phi_I = J \sum (1 + \sigma_t \sigma_{t+1}) \tag{6.1}$$

while the external field energy is

$$\phi_u = \sum_{x' = x_t}^{\infty} u(x', t) \tag{6.2}$$

of course with

$$\sigma_t = x_{t+1} - x_t, \qquad x_t = \sum_{0}^{t-1} \sigma_s \tag{6.3}$$

This model has indeed been solved,(11) using the (x_t, σ_t) representation. However, the phenomena occurring are seen most easily in the continuous "drum-head" model(12) obtained by appropriate scaling. Here, if $\xi(t)$ denotes the horizontal excursion of the interface at height t, then the microscopic fluid density at the point (x, t) is given by

$$n_\xi(x, t) = \varepsilon(x - \xi(t)) \tag{6.4}$$

If we define

$$U'(x, t) = \mu - u(x, t) \tag{6.5}$$

then for unit surface tension, the energy of configuration $\{\xi(t)\}$ is $[\dot{\xi} = d\xi/dt,\ U'(x, t) = \partial U(x, t)/\partial x]$

$$E[\xi] = \int \left[\tfrac{1}{2}\dot{\xi}(t)^2 + U(\xi(t), t)\right] dt \tag{6.6}$$

assuming small interface slope with respect to the vertical to justify the quadratic surface energy. Choosing $kT = 1$, we see that the grand partition function is now the path integral

$$\Xi = \int e^{-E[\xi]} D\xi(t) \tag{6.7}$$

If we introduce the truncated partition functions

$$\begin{aligned} \Xi(x, t) &= \int_{(0,0)}^{(x,t)} e^{-E[\xi]} D\xi(t) \\ \hat{\Xi}(x, t) &= \int_{(x,t)}^{(0,A)} e^{-E[\xi]} D\xi(t) \end{aligned} \tag{6.8}$$

where the interface is regarded as pinned at the indicated points, we then have

$$n'(x, t) = \Xi(x, t)\, \hat{\Xi}(x, t)/\Xi \tag{6.9}$$

The path integrals (6.8) are recognized as satisfying the Bloch equations

$$\dot{\Xi} = \tfrac{1}{2}\Xi'' - U\Xi, \qquad -\dot{\hat{\Xi}} = \tfrac{1}{2}\hat{\Xi}'' - U\hat{\Xi} \tag{6.10}$$

and it is routine to eliminate Ξ and $\hat{\Xi}$ from (6.9) and (6.10). There results the profile equation

$$\mu - u = \frac{1}{2}\left(\frac{\dot{n} + \frac{1}{2}n''}{n'}\right)^2 - \left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\left(\frac{\dot{n} + \frac{1}{2}n''}{n'}\right) \tag{6.11}$$

which integrates without difficulty to create the internal free energy$^{(13)}$

$$\bar{F} = \frac{1}{2}\iint \left(\frac{\dot{n}^2}{n'} + \frac{1}{4}\frac{n''^2}{n'}\right) dx\, dt \tag{6.12}$$

The point is now clear: $\bar{F}$ is indeed a local free energy, with a free energy density containing only finite derviatives of n, *but* n' occurs in the denominator, so that there is no way of expanding about a locally uniform reference.

7. CONCLUDING REMARKS

We have investigated the question of localized inverse response on Ising lattices by means of a number of exactly solvable model systems. This allowed us to show that sharp locality need not be broken by the existence of the fluctuations inherent in two-phase systems, but that the presence of an interaction circuit certainly suffices. An examination of the cooperativity of multiple interaction circuits is work for the future. We have also seen via an interface model that sharp locality does not imply the meaningfulness of local uniformity, but the validity—or modifications—of this statement in higher dimensionality is likewise an open question. It seems that such models are valuable tools for understanding the interaction between local and global behavior, and several more examples are in the process of evaluation.

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